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A Nonreflecting Plane Boundary for Wave Propagation Problems

WARWICK D. SMITH*

Seismographic Station, Department of Geology and Geophysics, University of California, Berkeley, California 94720

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Reflections from the boundary of a model may be completely eliminated by adding together the solutions of the Dirichlet and Neumann problems. The formulation is applicable to the solution of the scalar and vector wave equations, to dilatational elastic waves where mode conversion can occur, and to surface waves. When there is more than one component of displacement, the Dirichlet and Neumann conditions are applied to alternate components at the boundary. The technique is easily implemented for "finite element" or "finite difference" calculations.

1. INTRODUCTION

In numerical modelling of wave propagation problems the presence of artificial boundaries introduces spurious reflections which contaminate the solution. It is highly desirable to be able to eliminate these reflections and thus simulate an infinite medium. The problem can be overcome by constructing a model of sufficient size that the required solution is obtained before the reflections arrive. But this is not always feasible, as the model size is limited by the available computer storage. The experimenter must often make do with a smaller model and endeavor to ensure that his results are not affected by its small size. Lysmer and Kuhlemeyer [1] have formulated a system of dashpots at the boundary which damp out most of the reflections. The damping, though frequency dependent, is almost total for a wide range of incidence angles. Ang and Newmark [2] have used properties of the transmission of D'Alembert forces to develop a nonreflecting boundary. Their formulation gives a good approximation to the total elimination of reflections in most cases, but it is still only an approximation. Its ultimate justification is numerical. Approximate extrapolation schemes are commonly used in hydrodynamic problems ([6] and references therein).

In the following a technique for completely eliminating the reflections is

* On leave from Geophysics Division, Department of Scientific and Industrial Research, P.O. Box 8005, Wellington, New Zealand.

presented. The analytical formulation is exact, independent of both frequency and incidence angle. It involves the superposition of solutions and is thus much more costly than a single solution, but it allows solutions uncontaminated by reflections to be obtained when computer facilities prohibit the construction of a very large model.

2. Analytical Formulation

(a) *The Scalar Wave Equation*

This case is almost trivial. The equation of motion is

$$\partial^2 u/\partial t^2 = c^2 \nabla^2 u, \tag{1}$$

where u is the displacement and c the velocity. For a boundary at $x = 0$ (see Fig. 1) and incidence at angle i an incident plane wave of angular frequency ω and unit amplitude may be expressed as

$$u = \exp(i\omega/c)(x \cos i + y \sin i - ct). \tag{2}$$

The plane of Fig. 1 is that of the incident and reflected waves, and the normal. The coordinate system may be chosen appropriately. The reflected wave will then be of the form

$$u = A \exp (i\omega/c)(-x \cos i + y \sin i - ct). \tag{3}$$

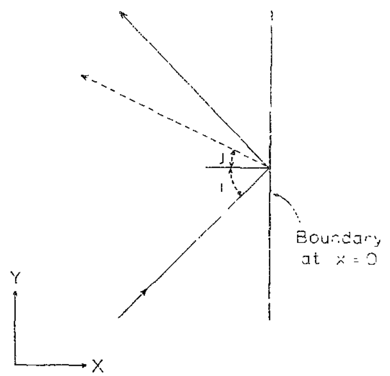

FIG. 1. The reflection scheme discussed in the text. The incident energy impinges at an angle i on the boundary ($x = 0$), and is reflected. The reflection denoted by the broken line is the SV wave which occurs when a P wave encounters a free boundary.

For a free boundary at $x = 0$, the Neumann condition $\partial u/\partial x = 0$ gives $A = 1$. For a fixed boundary, the Dirichlet condition $u = 0$ gives $A = -1$. Addition of the two solutions entirely eliminates the reflection. The same relationship holds if propagation is damped, since the imaginary exponents are replaced by complex ones, and the mathematics is otherwise unchanged. For curved wavefronts, one may take an image source which gives rise to the reflected wave. The Neumann condition requires the opposite sign. Addition eliminates the reflection.

(b) *Dilatational Elastic Waves and the Vector Wave Equation*

Consider first dilatational elastic waves. The vector wave equation is a special case. The equation of motion is

$$\rho(\partial^2\mathbf{u}/\partial t^2) = (\lambda + \mu)\,\nabla(\nabla\cdot\mathbf{u}) + \mu\nabla^2\mathbf{u}, \tag{4}$$

where ρ is the density, μ the rigidity, and λ the first Lamé parameter. The two solutions which will be added are the following:

Boundary Value Problem 1. The x displacement and the tangential stress are set to zero on the boundary. That is,

$$u_x = 0 \qquad \text{(Dirichlet for } u_x\text{)}, \tag{5}$$

$$p_{xy} = \mu[(\partial u_x/\partial y) + (\partial u_y/\partial x)] = 0. \tag{6}$$

Using (5), (6) becomes

$$\partial u_y/\partial x = 0 \qquad \text{at} \quad x = 0 \text{ (Neumann for } u_y\text{)}. \tag{7}$$

Boundary Value Problem 2. The y displacement and the normal stress are set to zero on the boundary.

i.e.

$$u_y = 0 \qquad \text{(Dirichlet for } u_y\text{)}, \tag{8}$$

$$p_{xx} = (\lambda + 2\mu)(\partial u_x/\partial x) + \lambda(\partial u_y/\partial_y) = 0, \tag{9}$$

hence

$$\partial u_x/\partial x = 0 \qquad \text{at} \quad x = 0 \text{ (Neumann for } u_x\text{)}. \tag{10}$$

Displacements may be expressed in terms of two potentials

$$u_x = (\partial\varphi/\partial x) - (\partial\psi/\partial y), \tag{11}$$

$$u_y = (\partial\varphi/\partial y) + (\partial\psi/\partial x). \tag{12}$$

The potential function φ describes longitudinal (P) motion, and ψ describes transverse (S) motion. The usual seismological terminology SV and SH will be adopted, to refer to S particle motion in the plane containing the propagation direction and the normal, and perpendicular to this plane, respectively. Section 2(a) above is applicable to SH motion.

The potentials for P incidence are

$$\varphi = \exp(i\omega/\alpha)(x\cos i + y\sin i - \alpha t) \quad \text{(Incident } P\text{)}$$
$$+ A\exp(i\omega/\alpha)(-x\cos i + y\sin i - \alpha t) \quad \text{(reflected } P\text{)}, \tag{13}$$

$$\psi = B\exp(i\omega/\beta)(-x\cos j + y\sin j - \beta t) \quad \text{(reflected } SV\text{)} \tag{14}$$

where $\alpha = P$ velocity, $\beta = S$ velocity, $i =$ incidence angle, and $j = SV$ reflection angle. Figure 1 shows the relevant geometry for incident P energy.

Solution 1

From (5) and (11)

$$(i\omega\cos i/\alpha)(1 - A)\exp(i\omega/\alpha)(y\sin i - \alpha t) = (i\omega\sin j/\beta)\,B\exp(i\omega/\beta)(y\sin j - \beta t). \tag{15}$$

$$\text{So } \sin i/\alpha = \sin j/\beta \text{ (Snell's Law)} \tag{16}$$

and

$$\cos i/\alpha\,(1 - A) = \sin j/\beta\,B. \tag{17}$$

From (7) and (12)

$$\begin{aligned} 0 &= (\partial^2\varphi/\partial x\,\partial y) + (\partial^2\psi/\partial x^2) \\ &= -(\omega^2\cos i\sin i/\alpha^2)(1 - A)\exp(i\omega/\alpha)(y\sin i - \alpha t) \\ &\quad - (\omega^2\cos^2 j/\beta^2)\,B\exp(i\omega/\beta)(y\sin j - \beta t); \end{aligned} \tag{18}$$

So

$$(\cos i\sin i/\alpha^2)(1 - A) = -(\cos^2 j/\beta^2)\,B. \tag{19}$$

From (17) and (19), $A = 1$ and $B = 0$. This corresponds to a P reflection in phase with the incident wave, and no SV reflection.

Solution 2

$$u_y = 0 \text{ gives } (\sin i/\alpha)(1 + A) = (\cos j/\beta)\,B, \tag{20}$$

$$\partial u_x/\partial x = 0 \text{ gives } (\cos^2 i/\alpha^2)(1 + A) = -(\cos j\sin j/\beta^2)B. \tag{21}$$

So $A = -1$ and $B = 0$. This implies a P reflection out of phase with the incident wave, and no SV reflection.

Therefore addition of the two solutions will exactly cancel the reflections. The case of incident SV waves is exactly parallel. It is a simple matter to show that there is no reflected P wave in either solution, and that the two reflected SV waves are of opposite sign. Addition eliminates all reflections. As for the scalar wave equation, the presence of a damping term does not affect the formulation.

In both the solutions, mode conversion is suppressed, so in fact the reflections are those for the vector wave equation. The same formulation therefore solves both the elastic case and the vector wave equation, such as would be applicable to acoustic waves in fluids, for example, or to electromagnetic wave propagation.

The two solutions can also be formulated in terms of image sources. P and SV reflections at a *free* boundary cannot be treated by image techniques, because of the mode conversion problem. The requirement that one displacement be zero, however, suppresses the conversion and allows the use of images. The image sources are of opposite sign in the two solutions, and they therefore disappear on addition.

(c) *Body Waves in Three Dimensions*

Since the SH component is uncoupled from the P and SV, it is still only necessary to form two solutions for each face. They are as follows.

BOUNDARY VALUE PROBLEM 1. Set the normal displacement on the boundary to zero.

BOUNDARY VALUE PROBLEM 2. Set the two displacements in the plane of the boundary to zero. The superposition of the two solutions will cancel all reflections.

(d) *Surface Waves*

It is perhaps not obvious that reflections of surface waves are also eliminated by this formulation. If Fig. 1 is now interpreted as a plan view of a three-dimensional model, showing the free surface, it can be used to illustrate the surface wave case. The wave encounters the boundary at an angle of incidence i, and is reflected.

(i) *Love waves.* Particle motion is horizontal, and transverse to the direction of propagation. At any particular depth, therefore, the elastic conditions to be satisfied at a boundary are exactly those for SV reflection in two dimension, i.e., transverse particle motion in the plane of the incident wave and the normal. As was indicated in (b), the addition of the two solutions cancels the reflection.

(ii) *Rayleigh waves.* Particle motion is in the vertical plane containing the propagation direction. At a free boundary there will be a reflected Rayleigh wave, together with body waves whose generation is necessary to satisfy the stress conditions at the boundary. It is beyond the scope of this paper to examine Rayleigh wave reflections in detail. Rather, it will suffice to show that if only the reflected Rayleigh wave is considered, reflections of opposite sign correspond to the Dirichlet conditions for normal and parallel displacements, respectively.

Potentials can be written as

$$\varphi = f(z)\{A \exp(i\omega/c)(x \cos i + y \sin i - ct) + A_1 \exp(i\omega/c)(-x \cos i + y \sin i - ct)\}, \tag{22}$$

$$\psi_x = g(z) \sin i\{B \exp(i\omega/c)(x \cos i + y \sin i - ct) + B_1 \exp(i\omega/c)(-x \cos i + y \sin i - ct)\}, \tag{23}$$

$$\psi_y = g(z) \cos i\{-B \exp(i\omega/c)(x \cos i + y \sin i - ct) + B_1 \exp(i\omega/c)(-x \cos i + y \sin i - ct)\}, \tag{24}$$

$$\psi_z = 0, \tag{25}$$

where

$$\mathbf{u} = \text{grad}\ \varphi + \text{curl}\ \boldsymbol{\psi}. \tag{26}$$

The functions $f(z)$ and $g(z)$, the phase velocity c, and the ratio B/A are given by the conditions that φ and $\boldsymbol{\psi}$ satisfy the wave equation with velocities $\alpha(z)$ and $\beta(z)$, respectively, and also that the stresses on the free surface are zero (see, e.g. [3]). The components of displacement at the boundary are

$$u_x = \frac{\partial\varphi}{\partial x} + \frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z} = \left\{\frac{i\omega \cos i}{c} f(z)(A - A_1) + g'(z) \cos i(B - B_1)\right\} \exp\frac{i\omega}{c}(y \sin i - ct), \tag{27}$$

$$u_y = \frac{\partial\varphi}{\partial y} + \frac{\partial\psi_x}{\partial z} - \frac{\partial\psi_z}{\partial x} = \left\{\frac{i\omega \sin i}{c} f(z)(A + A_1) + g'(z) \sin i(B + B_1)\right\} \exp\frac{i\omega}{c}(y \sin i - ct), \tag{28}$$

$$u_z = \frac{\partial\varphi}{\partial z} + \frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y} = \left\{f'(z)(A + A_1) - \frac{i\omega}{c} g(z)(B + B_1)\right\} \exp\frac{i\omega}{c}(y \sin i - ct). \tag{29}$$

A positive reflection ($A_1 = A$, $B_1 = B$) thus demands that $u_x = 0$, and a negative reflection demands $u_y = u_z = 0$. It is a simple matter to show that the stress requirements, which degenerate to the Neumann conditions as before, are also satisfied. It should be emphasized that this is not a complete proof for plane Rayleigh wave incidence, but it does indicate that reflections of Rayleigh waves might be expected to be canceled by exactly the same formulation as for body waves.

3. Multiple Reflections

When more than one face of the model is required to be nonreflecting, more solutions must be added to eliminate multiple reflections. Figure 2 illustrates the case of a corner, with incident *SV* energy. In Fig. 2a the displacements normal to the boundary have been held to zero at each face, and in Fig. 2b the displacements parallel to the boundary have been held to zero. The second order reflection is of the same sign in both cases, so two further solutions (2c and 2d) must be added in order to eliminate all reflections. The same group of four solutions is necessary for *P* wave incidence, and for the scalar wave equation case the four solutions are the combinations of the "fixed" and "free" boundary conditions. In general, if reflections are required to be eliminated on n surfaces 2^n solutions must be added. So eight solutions would be necessary at a three-dimensional corner.

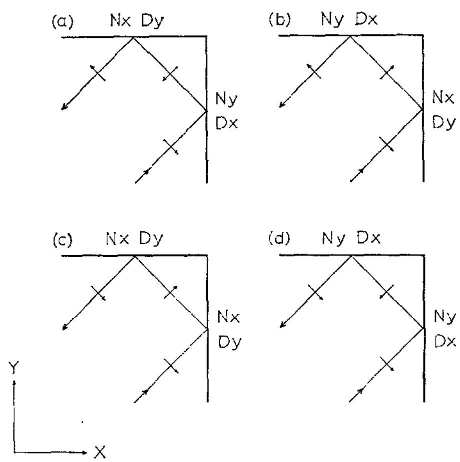

Fig. 2. The four solutions necessary for *P* or SV incidence at a corner. The particular case shown is the *SV* wave, and the small transverse arrows indicate the polarity at each stage. "*Dx*" implies the Dirichlet condition for the x component, "Ny" the Neumann condition for the y component, etc.

In practice it may not be necessary to add all the solutions to cancel the highest order reflections. These arrive later, so it may be possible to obtain the required information before they arrive. There are, in fact, high order reflections which cannot be cancelled. They can occur when a ray path encounters the same face more than once. An example is given later. These reflections are of such high order, and their travel time so long, that they are of little practical importance.

4. Generation of Surface Waves at Boundaries

When a body wave whose wavefront has large curvature encounters a free surface, surface waves can be generated. This is undesirable when the boundary is to be nonreflecting. The Rayleigh wave case is covered by the fact that none of the solutions to be combined has a free boundary. One or other of the displacements is always held to zero, and therefore Rayleigh waves can never propagate.

Love waves, however, can be generated by SH incidence, and one of the two solutions to be considered is the "free boundary" solution. No Love waves will propagate for the second solution ("fixed boundary") so there will be some residual when the sum is formed. Such waves will be noticeable only in the vicinity of the boundary, so it is not vital that they be eliminated, though it is certainly desirable. Their generation could be suppressed by ensuring that the model be homogeneous near the boundary. Love waves do not propagate in a homogeneous halfspace.

5. Numerical Implementation. A Finite Element Scheme in Two Dimensions

Propagation problems are conveniently treated by a dynamic "finite element" scheme employing the lumped mass matrix (see, e.g. [4]). This requires the solution of a matrix differential equation

$$[K]\,\mathbf{u} + [C]\,\dot{\mathbf{u}} + [M]\,\ddot{\mathbf{u}} = \mathbf{F}, \tag{3}$$

where $[K]$ is the stiffness matrix, $[C]$ is a damping matrix, $[M]$ is the diagonal mass matrix, and $\mathbf{F}$ is a forcing function. The vector $\mathbf{u}$ contains the displacements at all the nodes of the mesh. A dot denotes differentiation with respect to time. Equation (3) is conveniently solved by a Runge–Kutta algorithm (e.g. [5]) which requires that it be cast into the form

$$\ddot{\mathbf{u}} = [M]^{-1}(\mathbf{F} - [K]\,\mathbf{u} - [C]\,\dot{\mathbf{u}}). \tag{31}$$

Since $[M]$ is diagonal, the inverse is trivial to obtain. The Dirichlet condition may be imposed by setting to zero the terms of $[M]^{-1}$ that correspond to the particular components of the boundary displacements that are to be zero. If these displacements are set to zero initially, they are forced to remain zero throughout the entire solution. The stress condition is automatically satisfied by the finite element formulation.

Figure 3 shows a model which demonstrates this technique. It is simply an assemblage of 576 square elements, and was excited at the upper left hand corner by a square pulse. Four solutions were added to eliminate reflections on the right hand and lower faces. The displacements observed at the points marked in Fig. 3 are shown in the subsequent figures. The frequency characteristics of the model are determined by the size of its elements. Wave lengths shorter than the element length will be severely attenuated, so the square input pulse becomes considerably rounded. The finite element solution also contains a ringing phenomenon, at a frequency characterisitic of the size of the elements. For display purposes the time series of Figs. 4 and 5 have been low-pass filtered, and the ringing almost entirely removed.

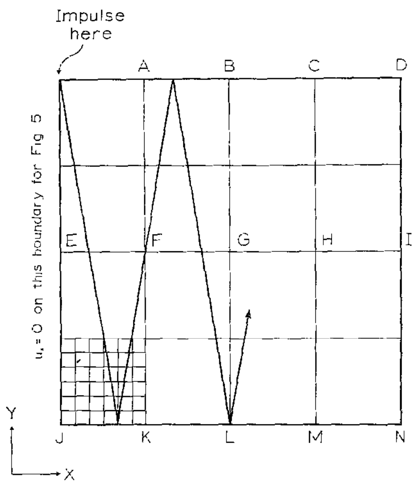

FIG. 3. The finite element model used to illustrate the technique. All elements are the same size, but fine detail of the model has been only partly indicated. The points A to N are those at which the displacements of Figs. 4 and 5 were observed. The ray path shown corresponds to one high order reflection which is not cancelled. The model is 1 km square, P and S velocities are 5.5 km/sec and 3.3 km/sec, respectively, and there is no damping.

Figure 4 illustrates a solution of the scalar wave equation. No damping or forcing terms (Eqs. (30), (31)) were used; the excitation was applied as a displacement pulse. The solution for free boundaries is shown in Fig. 4a, and in Fig. 4b can be seen the mean of the four solutions. The only reflection occurring in Fig. 4b, visible on traces D and I through N, corresponds to the ray path shown in Fig. 3, and to the corresponding path nearly parallel to the x axis. This reflection is not cancelled since it involves two encounters with a nonreflecting face, and so the reflected wave always has a positive sign. In practice this causes little difficulty, as models will in general be larger than that of Fig. 3, and this high order reflection will be delayed until after the required solution has been obtained. Its presence does imply, however, that it is not feasible to excite a model with a time series of very long duration. Rather, one should endeavor to obtain the impulse response, and then convolve with the desired input. If the model incorporates damping, the high order reflections may be damped out completely, and this would of course allow excitation by a long time series, without contamination from reflections.

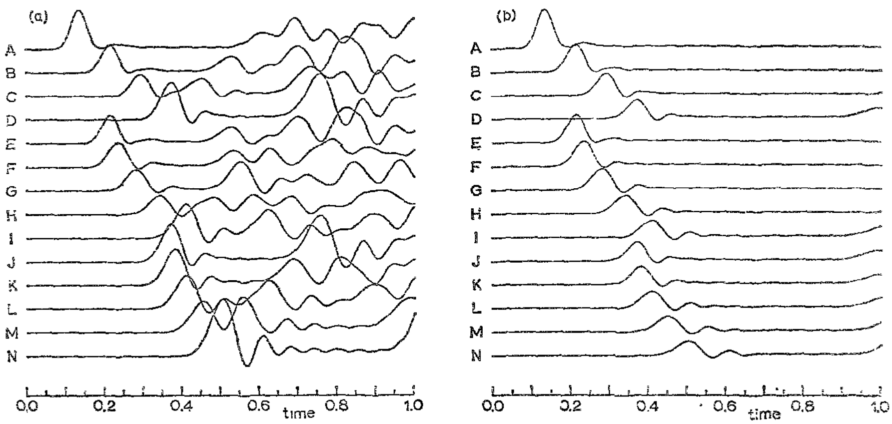

FIG. 4. Observed displacements for the scalar wave equation, i.e., *SH* motion in two dimensions. In Fig. 4a no effort has been made to cancel reflections. Fig. 4b represents the sum of the four solutions necessary to eliminate reflections on the lower and right hand faces of the model. Time is in seconds.

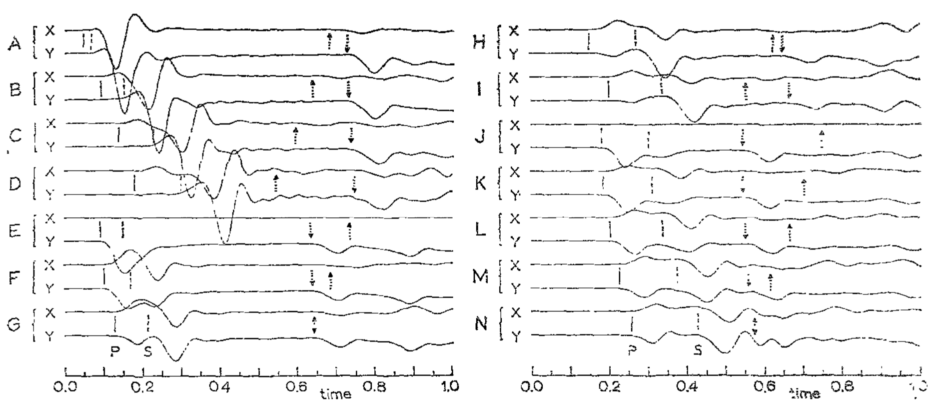

FIG. 5. The *P–SV* case, showing the *x* and *y* components of motion at each point. Four solutions have been added, as in Fig. 2. Theoretical arrival times for *P* (solid line) and *S* (broken line) are shown, and also for the reflection corresponding to the ray path of Fig. 3 (dotted line). The large displacements following the *S* wave on the free surface are those of the Rayleigh wave.

Figure 5 shows the mean of the four solutions for the dilatational case. The square exciting pulse was applied in the negative y direction, and the x displacements along the left-hand face of the model were held to zero to prevent Rayleigh waves propagating down that face. The Rayleigh pulse can be seen clearly in traces A–D, as the large displacements following the S pulse. No reflected Rayleigh wave returns from D. Calculated arrival times as shown in the figure identify the P and S pulses and the third order reflections of Fig. 3, from the right hand and bottom faces of the model. The latter reflections are visible mainly on the y components of displacement, and are stronger because P wave radiation is strongest in the y direction. Reflections from the right-hand face are detectable on the x component. The arrows in the figure indicate the appropriate component where the reflection may be seen. There are also some subsequent reflections which cannot be eliminated, but an increase in the size of the model will delay their arrival. It should be appreciated that doubling the size of the model delays by a factor of eight the time of first detection of unwanted reflections.

This finite element formulation is of course only one way of implementing the proposed technique. There are others, some possibly more efficient, but it serves to demonstrate the rather dramatic elimination of reflections. There is no difficulty in incorporating the scheme into "finite difference" calculations.

6. Oblique and Curved Boundaries

Thus far only plane boundaries, parallel to one of the coordinate axes, have been treated. It would be convenient if the formulation could be extended to plane boundaries inclined to the axes, and to curved boundaries. For inclined plane boundaries, it is necessary to express the displacement at the boundary in terms of components normal and parallel to the boundary face, rather than parallel to the coordinate axes. This can be accomplished for finite element formulations by a simple transformation matrix, and no doubt can also be implemented for finite difference schemes. The problem of curved boundaries presents a further difficulty, namely that if the boundary is concave to the incident wavefront, multiple reflections can occur. It therefore seems advisable to use plane boundaries, unless there are compelling reasons to introduce curvature.

7. Conclusions

Reflections may be eliminated in wave propagation problems by superimposing solutions which satisfy the Dirichlet and Neumann boundary conditions, respectively. In cases where there is more than one component of displacement, these conditions are applied to alternate components at the boundary. If n boundary faces are required to be nonreflecting, 2^n solutions will be necessary for a full solution, although it is rarely necessary to compute all these. Certain reflections involving multiple encounters with the same boundary face of a model cannot be eliminated, but this has only limited effect since they may be delayed by increasing the size of the model, or removed completely by incorporating damping. The technique is easily implemented for finite element or finite difference schemes and is applicable when particle motion is governed by the scalar or vector wave equation, or in the more general elastic case, including surface waves. The presence of damping terms does not alter the formulation.

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